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Binary Tree Approach to Scaling in Unimodal Maps

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Ge, Rusjan, and Zweifel introduced a binary tree which represents all the periodic windows in the chaotic regime of iterated one-dimensional unimodal maps. We consider the scaling behavior in a modified tree which takes into account the self-similarity of the window structure. A nonuniversal geometric convergence of the associated superstable parameter values towards a Misiurewicz point is observed for almost all binary sequences with periodic tails. For these sequences the window period grows arithmetically down the binary tree. There are an infinite number of exceptional sequences, however, for which the growth of the window period is faster. Numerical studies with a quadratic maximum suggest more rapid than geometric scaling of the superstable parameter values for such sequences.

KEY WORDS: Unimodal map; scaling; binary tree; periodic window; chaos; Misiurewicz point.

1. INTRODUCTION

Iterated one-dimensional unimodal maps[1] have been the subject of hundreds of papers over the last couple of decades. Not only are they among the simplest dynamical systems exhibiting chaotic behavior, but they are also very important as prototypes of dissipative systems. In spite of breakthroughs such as symbolic dynamics,[1-3] ergodic behavior,[1,4,6],[3] and the transition to chaos[7] in the theory of unimodal maps, many problems remain related to the scaling behavior of such maps within the so-called chaotic regime. This region is defined as the parameter interval between the

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first period-doubling accumulation point and the final crisis point beyond which no periodic or chaotic attractors can be found within the unimodality interval of the phase space.

Rigorous mathematical proofs[6] establish that the parameter values corresponding to an absolutely continuous invariant ergodic measure form a set with a positive Lebesgue measure. These "chaotic" parameter values are found in between the infinite number of windows with stable periodic attractors. The "periodic" windows are expected to be dense on the parameter axis. Although each window has a finite length, there remains a great deal of space for chaotic parameter values: Near the accumulation point of a period-doubling cascade, the relative fraction of the aperiodic solutions is given by the universal number 0.892... .[8] Even considering the whole chaotic region, the probability of finding an aperiodic solution is approximately 9/10 for a typical map.[9]

Since periodic windows are ubiquitous along the parameter axis, various infinite sequences of them are a natural tool when investigating scaling properties of unimodal maps. The scaling behaviors of period-doubling[7] and more general multifurcation sequences,[10] period-adding sequences approaching a crisis,[11,12] and tangent bifurcation points[13] have been determined. Shibayama[14] extended the analysis to the so-called Fibonacci sequences whose scaling is superexponential both on the parameter axis and in the phase space. An exact universal form of that type of scaling was found by Ketoja and Piirilä[15] using a renormalization argument. Later Lyubich and Milnor[16] derived rigorous results both for the scaling behavior and the dynamics at the accumulation points of such sequences.

In this paper we report scaling results related to a binary tree of periodic windows introduced by Ge *et al.* (GRZ).[17] Originally the tree was defined so that each window included the period-doubling tail in addition to the stable parameter interval of a periodic solution. In our modified tree, each window is extended up to the corresponding interior crisis point. We "sum" not only over the period-doubling tail, but over all the multifurcation sequences. In this way the self-similarity of the periodic-window structure[18] can be naturally taken into account. The structure within each window is essentially a small copy of the entire structure along the parameter axis. We concentrate on those periodic-window sequences whose binary codes have periodic tails. Such sequences usually lead to arithmetic growth of the period and to nonuniversal geometric scaling. The windows accumulate at a Misiurewicz point,[1,4] at which the dynamics is completely chaotic. The previously studied Fibonacci sequences form an exception with geometric growth of the period down the binary tree. We devise a method of generating an infinite number of other exceptional cases in which the period increases in a nonarithmetic way. Preliminary numerical studies suggest that the scaling in these cases is faster than geometric provided the critical point of the map is quadratic.

The rest of the paper is organized as follows. In Section 2 we discuss the window structure and the Metropolis *et al.* (MSS) rule[2] and explain our binary tree which keeps track of only the essential basic windows without copies induced by the self-similar structure. In Section 3 a new rule, different from MSS, is introduced for passing from a binary code to the MSS sequence. It is demonstrated in Section 4 that the new rule allows one to understand how a small fraction of the binary codes with periodic tails can lead to MSS sequences with geometrically growing lengths. Section 5 reports on the scaling of the positions and the widths of the windows belonging to typical, exceptional, or aperiodic binary codes. Section 6 generalizes a relationship of Post and Capel[12] between the scalings of the positions and the widths of the windows. We show that their result applies to all families corresponding to typical binary codes with periodic tails. There is a concluding discussion section, Section 7. An appendix contains the formal proof that the accumulation points of windows corresponding to typical binary codes with periodic tails are Misiurewicz points.

2. BINARY TREE

Consider a one-parameter family $f_\mu(x)$ of differentiable unimodal maps from a real interval I to itself. f_μ is assumed to have a quadratic maximum at $x = c$ so that the map is monotonically increasing for $x \in I$, $x < c$, and monotonically decreasing on the other side of the critical point c. An orbit obtained by iterating f_μ starting from c can be symbolically represented in terms of the kneading sequence $a_1 a_2 \ldots$, where $a_i = R$ if $f^i_\mu(c) > c$ and $a_i = L$ if $f^i_\mu(c) < c$. The case in which the orbit returns back to the critical point after i iterations, $f^i_\mu(c) = c$, is indicated by cutting the kneading sequence after $i-1$ symbols so that the sequence becomes finite. Finite symbol sequences therefore correspond to superstable periodic orbits. We are interested in the admissible kneading sequences at some parameter values of the unimodal map. Metropolis *et al.* (MSS)[2] discovered a simple rule by which all admissible symbol sequences, the so-called MSS sequences, can be generated and arranged on the parameter axis. Originally the rule was developed with one-parameter families of one-dimensional maps in mind, the type where the parameter appears as a multiplicative factor in the definition of the map. It holds, nevertheless, in a much larger class of unimodal maps, e.g., the logistic map $f_\mu(x) = \mu - x^2$.

It is instructive to take a look at the origin of the rule. To this end, consider two parameter values μ_1 and μ_2 ($\mu_1 < \mu_2$) which correspond to the infinite MSS sequences A and B. As in ref. 17, the beginning shared by the sequences is denoted by $A \wedge B$. In other words, μ_1 corresponds to the symbolic orbit $(A \wedge B)\, a_n \ldots$ and μ_2 to the orbit $(A \wedge B)\, b_n \ldots$, where the symbols a_n and b_n differ. The parameter dependence is assumed such that the MSS sequences in the interval (μ_1, μ_2) always begin with $A \wedge B$. By continuity there has to be a parameter value $\mu_3 \in (\mu_1, \mu_2)$ with the finite symbol sequence $C = A \wedge B$. The corresponding orbit is superstable with the period n. Within the interval (μ_1, μ_2), there are no other orbits with periods $\leqslant n$. Above μ_3, but within the window for the stable period n, the MSS sequence has the form $h(C) = Cb_nCb_nCb_n \ldots$ (the nth symbol has to be b_n, otherwise there would be at least two parameter values with the MSS sequence C). From $h(C)$ and B a new periodic window in between μ_3 and μ_2 can be constructed. On the other hand, the MSS sequence has the form $a(C) = Ca_nCa_nCa_n \ldots$ within the period-n window below μ_3. Now the infinite sequences A and $a(C)$ can be chosen to construct a periodic window in between μ_1 and μ_3. In this way a recursive procedure of generating new periodic windows is obtained. In order to generate MSS sequences for aperiodic orbits one would have to repeat the procedure an infinite number of times.

One needs two MSS sequences in order to get the procedure going. For a so-called full one-parameter family of unimodal maps,[1] such as the logistic map, one can set out with the symbol sequences for the superstable period-two cycle (R) and the final crisis point $(RLLL \ldots)$. In the latter, c is mapped onto an unstable fixed point after two iterations. The beginning shared by $h(R) = RLRLRL \ldots$ and $RLLL \ldots$ is RL, the MSS sequence for the first new periodic window. The next "level" becomes $h(R) \wedge a(RL) = RLR$ and $h(RL) \wedge RLLL \ldots = RLL$. However, as $h(R) = a(RLR)$, no new windows in between R and RLR can be constructed. R and RLR describe two consecutive windows within the same period-doubling sequence. According to this example it is impossible to generate an infinite binary tree of periodic windows using the original MSS rule. It will be shown below how to get around this problem.

The infinite sequences $h(C)$ and $a(C)$ above are called the harmonic and antiharmonic extensions of C. These extensions can be expressed without knowing the infinite "parent" sequences A and B. Let us first write the extensions in the form

$$h(C) = C\alpha C\alpha C\alpha \ldots$$

$$a(C) = C\bar{\alpha} C\bar{\alpha} C\bar{\alpha} \ldots$$

where $\bar{\alpha}$ denotes the "conjugate" of the symbol α; i.e., $\bar{R} = L$ and $\bar{L} = R$. In the sequel we will refer to α or $\bar{\alpha}$ as binding elements of an extension.

Around the critical point c, there is a small box in which the nth iterate of the unimodal map looks very similar to the first iterate. The nth iterate has either a maximum or a minimum at c. At the superstable parameter value with the MSS sequence C, the extremum touches the critical point. As the value of the parameter is increased within the stability interval, the extremum passes either above or below the critical point, depending on whether the extremum is a maximum or a minimum. If C contains an even number of R's (C even), the nth iterate has a maximum at c; otherwise (C odd) the extremum is a minimum.[12] In other words, $\alpha = R$ in the former and $\alpha = L$ in the latter case.

The fact that the nth iterate restricted to a small box around the critical point becomes a unimodal map is responsible for the self-similarity of the MSS structure. One expects the same MSS periodic windows with the nth iterate as with the original map. Only the structure in the higher iterate is observed in a much smaller parameter interval than for the original map. The window with the "reduced" MSS sequence $a_1 \ldots a_k$, which includes only every nth iterate (these actually land inside the small box), corresponds to the full MSS sequence[18]

$$C * (a_1 \ldots a_k) = \begin{cases} Ca_1 Ca_2 \ldots Ca_k C & \text{if } C \text{ is even} \\ C\bar{a}_1 C\bar{a}_2 \ldots C\bar{a}_k C & \text{if } C \text{ is odd} \end{cases}$$

for the original map. By this composition law it is easy to write down the MSS sequence at the endpoint of the parameter interval which contains the self-similar copy of the whole periodic window structure: $C * (RLLL\ldots) = C\alpha C\bar{\alpha} C\bar{\alpha} \ldots = C\alpha a(C)$. This endpoint corresponds to an internal crisis where the orbit of the critical point lands on an unstable period n after $2n$ iterations. We call $C\alpha a(C)$ the crisis extension and denote it by $e(C)$.

The superstable period-two cycle with the MSS code R, preceded by a stable fixed point on the parameter axis, belongs to the primary period-doubling cascade which ends at the transition to chaos. By self-similarity, every periodic window is followed up by a similar cascade. GRZ[17] modify the definition of a periodic window, including in it the corresponding period-doubling tail. In the recursive construction of the periodic windows one considers, instead of $h(C)$, the MSS code at the period-doubling accumulation point, $\hat{h}(\hat{h}(\hat{h}(\ldots \hat{h}(C)\ldots)))$, where the "cut" harmonic extension $\hat{h}(C) = C\alpha C$ is successively applied an infinite number of times. In this way, one never generates windows within a period-doubling cascade and two neighboring infinite sequences never become equal. Therefore, it is possible to generate an infinite binary tree of periodic windows. GRZ want to apply Feigenbaum's[19] general ideas on the renormalization of binary trees to this particular case. We take a different point of view and exploit the binary tree as a tool of studying the overall scaling behavior within the chaotic region.

We modify the GRZ tree with the crisis extension so as to take full advantage of the self-similarity of the window structure. Each stability interval is followed up by a self-similar copy of the whole periodic window structure. Because the scaling of these small "subwindows" is expected to be qualitatively equivalent to the scaling of the main structure, we omit them by extending each window up to the corresponding interior crisis point. Furthermore, by self-similarity the scaling within the 2^p-band regions ($p = 1, 2, \ldots$)[1] is expected to be similar to the one observed within the region of one chaotic band. We consider only the latter region and construct an infinite binary tree of periodic windows making use of the antiharmonic and crisis extensions in the following way:

1. Begin with the infinite "left parent" $e(R)$ and the infinite "right parent" $RLLL\ldots$. The crisis extension $e(R) = RLRRR\ldots$ corresponds to the last band merging point (where a critical point is mapped onto an unstable fixed point after three iterations) and the sequence $RLLL\ldots$ to the final crisis point. All the windows of the tree lie in between these two points (i.e., in the region of one chaotic band[1]).

2. From two infinite "parent" sequences A and B form a finite "daughter" sequence C by taking the shared beginning of A and B. The first such sequence RL is the "root" of the tree.

3. Take A and $a(C)$ as the infinite parents of a new left "branch" and $e(C)$ and B as the infinite parents of a new right "branch." Attach the symbol 0 to the left and the symbol 1 to the right branch.

The beginning of the infinite binary tree is shown in Fig. 1. There is a one-to-one correspondence between the binary codes consisting of the symbols 0 and 1 and the MSS sequences. In the following, we use the symbol $\rightarrow$ to express this correspondence. For example, $10^2 \rightarrow RL^2RLR$. α^k means that the symbol (or a block of symbols) α is repeated k times. This convention is used for both the MSS sequences and the binary codes.

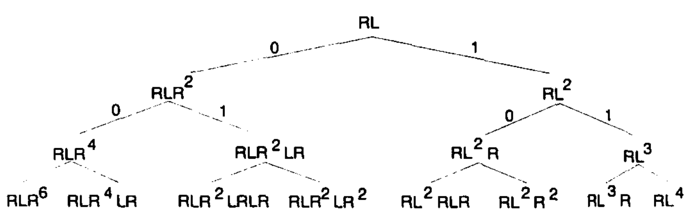

Fig. 1. The beginning of the infinite binary tree of periodic windows.

3. TRANSFORMATION BETWEEN THE BINARY CODE AND THE MSS SEQUENCE

In this section we develop "self-contained" recursive rules by which the transformation $\rightarrow$ can be carried out. These rules are just another variant of the MSS rule, but they turn out to be the key to understanding how the length of the MSS sequence increases down the binary tree. In the following, the ith symbol of the MSS sequence A is denoted by $\{A\}_i$ and the string from the ith symbol up to the jth symbol by $\{A\}_i^j$ ($j<i$ implies an empty string). This notation is particularly useful if A is an extension or some other composition.

Assume that an infinite binary code $i_1 i_2 \ldots$ corresponds to the MSS sequence A. Let A_k be the truncation of A so that $i_1 i_2 \ldots i_k \rightarrow A_k$. One of the "parent" branches of A_{k+1} is always A_k. The more distant parent of A_{k+1} is denoted by $\hat{A}_{k+1}$ (and that of A_k is $\hat{A}_k$). For example, the parent branches of $A_3 = RL^2RLR$ are $A_2 = RL^2R$ and $\hat{A}_3 = RL$. $\hat{A}_k$ can be defined also for the case in which A_k lies at the edge of the binary tree (see below). The infinite parents of A_{k+1} are either $a(A_k)$ and $e(\hat{A}_{k+1})$ or $e(A_k)$ and $a(\hat{A}_{k+1})$. It is then clear that A_{k+1} can be written either in the form

$$A_{k+1} = A_k \beta_k \{a(A_k)\}_1^{h_k}$$

or

$$A_{k+1} = \hat{A}_{k+1} \gamma_{k+1} \{a(\hat{A}_{k+1})\}_1^{m_{k+1}}$$

where β_k and γ_{k+1} are the first binding elements of the proper extension. β_k, h_k, and m_k are needed in the construction of the MSS sequence and can be determined by the following rules:

Rule 1. (a) If $i_{k+1} = i_k$, then $\beta_k = \{a(\hat{A}_k)\}_{1+m_k}$ and h_k is the length of the sequence

$$a(A_k) \wedge \{a(\hat{A}_k)\}_{2+m_k}^{\infty}$$

(b) $m_{k+1} = m_k + h_k + 1$ and $\hat{A}_{k+1} = \hat{A}_k$.

Proof. With $i_{k+1} = i_k$ the same parent branch is approached as on the previous step, so that $\hat{A}_{k+1} = \hat{A}_k$. The immediate infinite parent of the new daughter branch A_{k+1} has the form $A_k \beta_k a(A_k)$, and the more distant infinite parent has the form $\hat{A}_k \gamma_k a(\hat{A}_k)$. A_{k+1} is their shared beginning. It can be longer than $A_k = \hat{A}_k \gamma_k \{a(\hat{A}_k)\}_1^{m_k}$ only if $\beta_k = \{a(\hat{A}_k)\}_{1+m_k}$. In other words,

$$A_{k+1} = A_k \beta_k [a(A_k) \wedge \{a(\hat{A}_k)\}_{2+m_k}^{\infty}]$$

Rule 1(b) can be easily verified by considering the lengths of the sequences in the above two equations for A_{k+1}, the latter also with the index k. ∎

Rule 2. (a) If $i_{k+1} \neq i_k$, then $\beta_k = \{a(A_{k-1})\}_{1+h_{k-1}}$ and h_k is the length of the sequence

$$a(A_k) \wedge \{a(A_{k-1})\}_{2+h_{k-1}}^{\infty}$$

(b) $m_{k+1} = h_{k-1} + h_k + 1$ and $\hat{A}_{k+1} = A_{k-1}$.

Proof. $i_{k+1} \neq i_k$ entails a turning back toward the $(k-1)$th branch. Thus, $\hat{A}_{k+1} = A_{k-1}$. The infinite parents of the new daughter branch have the forms $A_{k-1}\beta_{k-1}a(A_{k-1})$ and $A_k\beta_k a(A_k)$. The shared beginning of the infinite parents can be longer than $A_k = A_{k-1}\beta_{k-1}\{a(A_{k-1})\}_1^{h_{k-1}}$ only if $\beta_k = \{a(A_{k-1})\}_{1+h_{k-1}}$. Thus,

$$A_{k+1} = A_k\beta_k[a(A_k) \wedge \{a(A_{k-1})\}_{2+h_{k-1}}^{\infty}] \quad ∎$$

In fact, it would suffice to memorize A_k, h_{k-1}, and m_k because $\hat{A}_k$ and A_{k-1} can be determined from these. If $i_1 = 0$, one has the initial conditions $A_1 = RLRR$, $h_0 = 1$, and $m_1 = 2$ $(\hat{A}_1 = R)$. If $i_1 = 1$, then $A_1 = RLL$, $h_0 = 0$, and $m_1 = 2$ $(\hat{A}_1 = \varnothing)$.

4. TYPICAL AND EXCEPTIONAL BINARY CODES

The proofs of Rules 1 and 2 involve only a slight elaboration on the MSS rule. The generation of the MSS sequence with these rules is not necessarily much more efficient than applying the MSS rule directly. The new formulation helps one understand why the length of the MSS sequence increases arithmetically with some binary codes and faster with others. The increasing length of the MSS sequence in a single binary step is given by $h_k + 1$. We call the growth of the sequence length arithmetic if $\sup\{H_1, H_2, \ldots\} < H < \infty$, where

$$H_K = \frac{1}{K} \sum_{k=1}^{K} h_k$$

This definition allows arbitrarily large occasional increments in the MSS sequence, but they may not be frequent.

h_k must become large for the MSS string length to grow rapidly. Let us denote the length of $\hat{A}_{k+1}$ by $l(\hat{A}_{k+1})$ and assume that $m_{k+1} < l(\hat{A}_{k+1})$

(call it the simple extension condition). Then both A_k and $\hat{A}_k$ [Rule 1(a)] or A_k and A_{k-1} [Rule 2(a)] are long enough so that we can replace the antiharmonic extensions with the infinite MSS sequence A in determining h_k. This suggests that h_k can be large only if there is a large block in A identical with the beginning of A and which lies after the first $1+m_k$ $(i_{k+1}=i_k)$ or $1+h_{k-1}$ $(i_{k+1}\neq i_k)$ symbols of A. On the other hand, the beginning of A can be written in the form $A=\hat{A}_j\gamma_j\{a(\hat{A}_j)\}_1^{m_j}\ldots$ anywhere in the sequence (at arbitrary j). Note that $a(\hat{A}_j)$ can be replaced by A if $l(\hat{A}_j)>m_j$. Thus m_j gives a rough estimate of the length of the above-described block, identical with the beginning of A, which comes following the first $l(\hat{A}_j)+1$ symbols. If m_k $(i_{k+1}=i_k)$ or h_{k-1} $(i_{k+1}\neq i_k)$ becomes equal to $l(\hat{A}_j)$ for some j for which m_j is large, then h_k can reach a large value. If such an index j is not found, then h_k can be expected to remain "small."

The initial values h_0 and m_1 are small, and Rule 2(b) implies that, taking steps in alternating directions in the binary tree, one is not likely to generate large values of m_j. According to Rule 1(b), m_j grows at least linearly with j if consecutive steps are taken in the same direction. In other words, assuming that $I(1-i)\, i^p \to \hat{A}_j\gamma_j\{A\}_1^{m_j}$ with $I\to\hat{A}_j$, we obtain $m_j\geqslant p$. In this way we see that arbitrarily large values of m_j and, accordingly, of h_k are possible. However, this is not sufficient to generate geometric growth of an MSS sequence. Geometric growth requires average unbounded increases of h_k as a function of k. It turns out that judiciously placed blocks of identical symbols can bring about such a phenomenon. It is difficult to explain the geometric growth of the MSS sequence length for an arbitrary binary code. In the following we consider a special form of the binary code whose growth properties are easier to understand.

Proposition. Consider binary codes of the form $1010^{p(2)}10^{p(3)}\ldots10^{p(n)}$. If $p(i)>0$ and $p(i)+j(i)<i$ $(i=2, 3,\ldots, n)$, where $j(i)$ is calculated from Eqs. (1)–(5) below, then the corresponding MSS sequence has the form

$$\alpha_{-1}[\cdot]_{-1}\,\alpha_0[\cdot]_0\cdots\alpha_n[\cdot]_n$$

with α_n either R or L according to the rule $\alpha_n=\alpha_{k(n)}$ $[k(n)<n]$ beginning with $\alpha_{-1}=R$ and $\alpha_0=L$ and

$$[\cdot]_n=[\cdot]_{k(n)}\,\alpha_{j(n)}[\cdot]_{j(n)}\,\alpha_{1+j(n)}[\cdot]_{1+j(n)}\cdots\alpha_{m(n)}[\cdot]_{m(n)}$$

Let $l(n)$ be the length of the block $[\cdot]_n$ and $s(n)$ the length of the whole

MSS sequence. Then $k(n)$, $j(n)$, $m(n)$, $l(n)$, and $s(n)$ are given by the recursion formulas

$$s(k(i)-1)=l(m(i-1)) \tag{1}$$

$$s(j(i)-1)=l(k(i)) \tag{2}$$

$$m(i)=j(i)+p(i)-1 \tag{3}$$

$$l(i)=s(m(i)) \tag{4}$$

$$s(i)=s(i-1)+l(i)+1 \tag{5}$$

with the initial conditions $k(1)=0$, $m(1)=-1$, $l(-1)=l(0)=0$, $l(1)=1$, $s(-2)=0$, $s(-1)=1$, $s(0)=2$, and $s(1)=4$. The MSS sequence begins with $RLL[R]_1 \ldots$.

The proof is a straightforward application of Rules 1 and 2 and is omitted here.

The condition $p(i)+j(i)<i$ is equivalent to the simple extension condition.

Equations (1)–(4) can be combined into a recursion rule for $m(n)$ alone:

$$m(n)=p(n)+m(m(m(n-1))+1) \tag{6}$$

In addition to the initial condition for $m(1)$ one has to specify the values $m(-1)=m(0)=-2$ in order to apply (6). The simple extension condition in terms of $m(i)$ becomes $m(i)<i-1$.

With the aid of the Proposition binary sequences can be constructed which lead to geometric growth of the MSS sequence length. Equations (4)–(5) give

$$s(n)=s(n-1)+s(m(n))+1 \tag{7}$$

for the length of the sequence. This implies that the window period grows according to the recursion $q_n=q_{n-1}+q_{m(n)}$ with each added block $10^{p(n)}$ in the binary code. The powers $p(n)$ in Eq. (6) can be chosen so that $m(n)=n-r$ with $r>1$ for $n>N$. The leading eigenvalue ζ of the transition matrix M defining the recursion via $(q_n, q_{n-1},\ldots, q_{n-r+1})=M(q_{n-1}, q_{n-2},\ldots, q_{n-r})$ is greater than unity and gives the asymptotic growth rate of the MSS sequence length. The simple extension condition implies $\zeta<2$.

All Fibonacci sequences correspond to $m(n)=n-2$ with $\zeta\approx 1.618$. Equation (6) implies that the power $p(n)$ takes the constant value 4 after the "transients" have died out.

Example. The MSS sequence for the binary code $1010^2 10^3 10^4 10^4 \ldots$ reads

$$RLL[R]_1 \, R[RL]_2 \, R[RLL(R)]_3 \, L[RLL(R)\, R(RL)]_4$$
$$\times L[(R)\, LL(R)\, R(RL)\, R(RLLR)]_5 \ldots$$

Each block $\alpha_n[\cdot]_n$ results from a block $10^{p(n)}$ in the binary code. The increments resulting from h different from zero are in parentheses $(\ldots)$. These increments are precisely the same as the blocks $[\cdot]_i$ and $m(i) = i - 2$ $(i = 1, 2, \ldots)$.

Examples of geometric growth with various r and asymptotic powers p are listed in Table I. All were originally found numerically, but the examples with $\zeta < 2$ are beautifully explained by the Proposition, which predicts correctly the binary period of families with the recursion $q_n = q_{n-1} + q_{m(n)}$, although their binary codes may not begin like 101.... The recursion $q_n = q_{n-2} + q_{n-3}$ can be put into this form by first replacing q_{n-3} by $q_{n-1} - q_{n-4}$ and then observing that $q_{n-2} - q_{n-4} = q_{n-5}$. More complicated cases can be constructed by letting $m(n)$ oscillate according to some rule. For example, taking $m(2n+1) = 2n - 1$ and $m(2n) = 2n - 3$ leads to the recursion relations $q_{2n+1} = q_{2n} + q_{2n-1}$ and $q_{2n} = q_{2n-1} + q_{2n-3}$, which can be combined as $q_{2n+1} = 2q_{2n-1} + q_{2n-3}$. From Eq. (6) one can solve for the powers $p(2) = 1$, $p(3) = p(4) = 3$, $p(5) = 5$, $p(2n) = 4$ $(n = 3, 4, \ldots)$, and $p(2n+1) = 6$ $(n = 3, 4, \ldots)$. Each asymptotic binary block $10^4 10^6$ means multiplying the period with the average factor $\zeta \approx 1.554$. Aperiodic binary codes can be generated making the oscillations in $m(n)$ aperiodic. Instead of letting $m(n)$ oscillate regularly as above, we took $m(n) = 2[n/2] - 3$ or $m(n) = 2[n/2] - 1$ at random ($[\cdot]$ stands for the integer part), beginning with an adjustable n. The longer the leading regular part, the faster the MSS sequences grew in length. On account of

Table I. Repeating Patterns of Periodic Binary Tails for Some Recursive Rules Leading to Asymptotic Geometric Increase (Given by Factor ζ) of the MSS Sequence Length

Rule	ζ	Pattern
$q_n = 2q_{n-1} + q_{n-2}$	2.414	10^{10}
$q_n = q_{n-1} + 2q_{n-2}$	2.000	10^3
$q_n = q_{n-1} + q_{n-2}$	1.618	10^4
$q_n = q_{n-1} + q_{n-3}$	1.466	10^6
$q_n = q_{n-1} + q_{n-4}$	1.380	10^8
$q_n = q_{n-2} + q_{n-3}$	1.325	10^{10}

inherent limitations of computers, no certain conclusion could be drawn about the asymptotic algebraic or geometric growth of such sequences. It may be interesting to notice that the sum of the second differences of the MSS lengths seldom seemed to display a sustained growth, suggesting an asymptotic algebraic fate to all MSS sequences arising from random period binary codes.

A purely numerical method of finding geometric growth consists of first defining a recursive rule for the increase of the window period and then determining the binary code which yields the windows (only a small number of rules lead to simple behavior). Each "appropriate" recursion rule has a characteristic repeating pattern of the periodic binary tail. For example, the rule $q_n = q_{n-1} + q_{n-2}$ gives the same repeating pattern 10^4 for all choices of q_1 and q_2 (if one always moves to the right in the binary tree). Taking off from an arbitrary window, however, and successively adding the pattern 10^4 to the binary code will almost certainly lead to asymptotically arithmetic growth of the MSS sequence length!

Geometric growth obviously requires a very synchronous binary code. A single mismatched binary symbol may suffice to turn the growth into arithmetic. Therefore, one expects the MSS sequence length to grow arithmetically for most binary codes with periodic tails. We call such codes "typical," whereas the codes leading to a faster growth of the MSS sequence are called "exceptional."

The examples of Table I have been constructed by always picking the closest matching window on the right-hand side. We could not find any exceptional codes with periodic tails by choosing the next window on the left-hand side. One may begin, nevertheless, with a binary code that carries a number of leading zeros.

5. NUMERICAL SCALING RESULTS

The scaling behavior of the positions and widths of the windows appears to be related to the manner of growth of the length of the MSS sequence along a path in the binary tree. In the following, the scaling of both typical and exceptional binary codes is discussed, concentrating on codes with periodic tails.

5.1. Typical Codes with Periodic Tails

The binary sequence 1^∞ corresponds to the stable periods 4, 5, 6,... approaching the final crisis point. The scaling properties of this sequence are well understood.[11,12] Along the parameter axis the scaling of the superstable parameter values is geometric. The scaling factor is determined by the derivative of the map at the unstable fixed point, which is the image of the critical point under the second iterate of the map. The widths of the windows scale by the square of this factor. Every other binary code with the tail 1^∞ corresponds to a window sequence approaching a tangent bifurcation point, i.e., the left end point of some periodic window. It has been shown[13] that this results in slower than geometrical scaling.

On the other hand, every binary sequence with the tail 0^∞ leads to a sequence which accumulates at an internal crisis point. All these points are fully chaotic according to a theorem by Misiurewicz.[1,4] In the same way as for the final crisis point, the scaling properties are determined by the Lyapunov factors of the associated unstable orbits (Section 6).

Let us now consider an arbitrary binary code of the form IJ^∞ and let μ_k and $\Delta\mu_k$ be the superstable parameter value of the window IJ^k and its width, respectively. We determine the scaling factors

$$\delta_k = \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}; \qquad \sigma_k = \frac{\Delta\mu_{k-1}}{\Delta\mu_k}$$

for the logistic map and find the period ν by which δ_k and σ_k oscillate as $k \to \infty$. The asymptotic limits of the products of ν subsequent δ_k's and σ_k's are denoted by δ and σ, respectively. Table II displays ν and δ for a number of examples. Neither ν nor δ is universally determined by the tail of the binary code. In Section 6 it is shown that both are related to the orbit of the critical point at the accumulation point of the window sequence. The

Table II. Scaling Period ν and Factor δ for Some Typical Binary Codes with Periodic Tails[a]

Code	ν	τ	δ	M_p	M_i
$(01)^\infty$	1	2	6.996	2	4
$(10)^\infty$	1	2	3.716	1	4
$0(01)^\infty$	1	2	5.560	2	6
$1(10)^\infty$	1	2	3.931	1	5
$(001)^\infty$	1	1	12.11	6	4
$(010)^\infty$	2	1	53.96	9	10
$(100)^\infty$	1	1	4.962	3	8
$(110)^\infty$	1	1	7.694	3	4
$(101)^\infty$	1	1	6.736	3	3
$(011)^\infty$	2	1	95.30	9	4

[a] M_p gives the period of the unstable orbit and M_i the length of the transient at the accumulation point. The repeating MSS pattern is gone through τ times during ν periods of the binary tail.

accumulation point for this class of codes is always a Misiurewicz point at which the critical point is mapped onto an unstable period M_p after M_i iterations. A proof of this claim is given in the appendix. The numbers M_p and M_i are included in Table II. It turns out that an MSS sequence corresponding to the typical binary code has a periodic tail, which is possible only if the critical point is mapped onto an unstable orbit. The repeating pattern gives the kneading sequence of the unstable orbit. The quantity ν is greater than unity if the whole pattern is not traversed in one period of the binary tail. It may happen that the pattern is run through more than once per binary period. In Table II, τ gives the number of times the pattern is completed in ν periods of the binary tail.

In Section 6 the result of Post and Capel[12] that $\sigma = \delta^2$ is generalized to all codes within this class. In other words, the scaling behavior in this class is equivalent to the one for the code 1^∞ and those with the tail 0^∞.

5.2. Exceptional Codes with Periodic Tails

It has been shown in the Fibonacci case that the scaling of the superstable parameter values is superexponential.[14,15] A geometric increase of the MSS sequence length leads to superexponential scaling in all the cases we studied. Furthermore, a comparison of the appropriate scaling factors[14,15] for the logistic map on the one hand and $f_\mu(x) = \mu \sin(\pi x)$ on the other suggests that the superexponential scaling is universal for quadratic-maximum maps.

It may be interesting to notice[20] that the scaling depends on the order of the maximum z. In particular, there seems to be a critical exponent z_c above which the scaling becomes geometric. This critical exponent appears to vary according to the rule determining the manner in which the MSS sequence length grows. z_c is known to be 2 for the Fibonacci sequences.[20] We illustrate this in Table III, where we give a number of consecutive values of the "second ratio" $(\mu_{n+1} - \mu_n)(\mu_{n-1} - \mu_{n-2})/(\mu_n - \mu_{n-1})^2$ as n increases. At the exponent $z = 1.9$ it can be seen that the second ratio becomes smaller, i.e., the scaling becomes tighter and tighter. At $z = 2$ the second ratios are approximately constant, and at $z = 2.1$ they grow and will eventually reach unity. Then the asymptotic scaling is geometric. The numerical result is not as clear with other rules of MSS length growth. As an example, take the rule number four in Table I, $q_n = q_{n-1} + q_{n-3}$. In Table IV we display the second ratios with this rule around what seems to be the critical exponent, $z = 3$. The general pattern is clear, as can be seen in tables of second ratios for values of z further away from $z = 3$ (not displayed here). Nevertheless, it is not obvious, contrary to the case of the Fibonacci rule, that one can determine the critical z accurately. It is even

Table III. Second Ratios $(\mu_{n+1}-\mu_n)(\mu_{n-1}-\mu_{n-2})/(\mu_n-\mu_{n-1})^2$ for Fibonacci Growth Rule of MSS Sequence Length with Three Different Values of z

$z = 1.9$
0.540568649228514548489190112723959
0.564530204503619057844967713704017
0.551578916360676664253283350185272
0.549726619390773431923899110249692
0.538351819176064005321551433050646
0.529614409340695420815325244595764
0.518136481799752541358716136148993
0.507141398531148030772618302788561
0.495330000671234148623798701714412
0.485857115865495547661709604051006
$z = 2$
0.576424419734741300857167214977898
0.614959130438622236460394486199093
0.613245632645603495309856166834155
0.623446012948384965895810527136745
0.623743192210949323918773767120082
0.627143470611499650661729361375020
0.627646394677352360059760018283382
0.628779540148719241232479892371548
0.629079829446494768497497089288172
0.629474761421065715816252235390039
$z = 2.1$
0.607806103957966838865150212339956
0.658870595830241944698609969133370
0.666284233714677552991319862655434
0.685565735311454240056896326101484
0.694426025960070510569799640832677
0.706458870619379371286744138097185
0.715214924771737555463117341141810
0.724415741171660654014734418540930
0.732511028043468910740220597152332
0.740481896079685679976317012804011

more difficult in other cases that we studied. Even if we knew that z_c has to be an integer, our results (with quadruple precision) would not warrant accurate claims about z_c for the rules $q_n = q_{n-1} + q_{n-4}$ or $q_n = q_{n-2} + q_{n-3}$ let alone $q_n = q_{n-1} + 2q_{n-2}$. What one can say in these cases is that the eventual scaling at large z will be geometric (exponential).

The suggestion about the universality of the superexponential scaling should be understood as follows. The Fibonacci rule is known to lead to

Table IV. Same as Table III, but for the Rule $q_n = q_{n-1} + q_{n-3}$

$z = 2.8$
0.586845054420196752379632215327061
0.690937115880963355221722871690154
0.598511221621835315814787239739078
0.680678918895344414159165613855318
0.630796289352418888415607504824941
0.650678698665798235063631683518628
0.622338978663477450589882400824111
0.636204265348214301337957145157083
0.616070279619805566300882847594128
0.617131414543638485422774548647918
$z = 2.9$
0.597854898135448817364366762193185
0.701122017794715060164718629128255
0.612233100582278084980439102737883
0.700396659628436265252161220149747
0.654472472610879111943123908355852
0.676791426764289821420866637369944
0.652604193388832420784901064991448
0.670040424124573012505170071979655
0.653900542597664328163447728516512
0.658379863517226631097254053686949
$z = 3$
0.889063230509710949257433103155827
0.608526128578811708253168260842363
0.710554728779540332908490734267502
0.624778128202789123792585376048476
0.718523854398598587334029966646016
0.676337976539586658795201438357151
0.700614931953453023999854784059066
0.680080751100011810567514638373817
0.700484168187937249113902119712843
0.687843759237454627198420807979781
0.695092414531686557839033507440895
0.690610650374720809239020313212933
0.695169218741026093691779912113938
$z = 3.1$
0.618879322690059536219019410471444
0.719300751880715595824307857228876
0.636240002730120893516230363300766
0.735185407285350370458877476608666
0.696529582120099118658733423994191
0.722350936087387120085389673522407
0.704996784254788709822982123279888
0.727819840994460703811864141746312
0.718208885237845542575719988558140
0.727646559370297053146699190784777
$z = 3.2$
0.628933320396211536160445480826680
0.727419159108398019398137074943382
0.646706236805181097598526489560081
0.750499007534655851165800338890541
0.715178663761871435238940601429359
0.742191453721296762234597144176702
0.727579330729961134500138727415663
0.752333864092286422175342622787652
0.745319953539087765336190212989834
0.756443669844986441757849531489885

a universal constant second ratio with all unimodal maps with $z=2$. We tested the logistic map and the sine map for a number of rules of growth. The computed second ratios varied in the same range as those for the Fibonacci rule with the two maps or with different beginnings of the Fibonacci chain. As the sine map has $z \doteq 2$, the conjecture of universality can also only concern $z=2$.

5.3. Codes with Aperiodic Tails

It was demonstrated in Section 4 that exceptional aperiodic codes with rapid growth rates of the MSS sequence length can be explicitly constructed. For a random code with no built-in blocks of identical symbols, however, it is natural to expect arithmetic average growth. The numerical studies bear out this expectation. We observe average geometric scalings of the superstable parameter values and positive Lyapunov exponents at the accumulation points.

6. GEOMETRIC SCALING

We are interested in families of MSS sequences belonging to typical binary codes with periodic tails. These sequences have some initial length M_i. They grow in steps of $n = \tau M_p$ per advancing ν periodic blocks of the binary tail. In order to get at the scaling parameters δ and σ defined in the preceding section, we take advantage of the local description of Post and Capel.[12] It describes the map $f_\mu^l(x)$ within the window of period l. We choose a particular representation of the logistic map $f_\mu(x) = \mu - |x|^z$ where z may take the values $1 < z < \infty$ (i.e., in this section we do not restrict ourselves to the case of a quadratic maximum). Then the local description is centered at $x=0$, and the following reduced form applies:

$$x_{l(i+1)} = \rho_l(\mu) + \lambda_l\, |x_{li}|^z + \kappa_l\, |x_{li}|^{2z} + \cdots \tag{8}$$

where $x_{l(i=0)}$ is some starting point close to the central value $x=0$, so close that we do not have to worry about the term to the power $2z$ on the right-hand side. The local description then incorporates one parameter λ_l. Finally

$$\rho_l(\mu) = \mu - |\mu - |\mu - \cdots |\mu|^z \cdots |^z|^z \tag{9}$$

with $l-1$ pairs of vertical bars. For the present purposes the index l will need to take values like $m + kn$ and $m + (k+1)n$, where k is a large integer, since we are interested in the asymptotic μ scaling and the ratios of widths of consecutive windows in our families.

With the substitutions, leaving out temporarily the subscripts of λ and ρ,

$$\begin{aligned} x_{li} &= u_i\,|\lambda|^{-1/(z-1)}\,\mathrm{sgn}\,\lambda \\ \rho &= -r\,|\lambda|^{-1/(z-1)}\,\mathrm{sgn}\,\lambda \end{aligned} \tag{10}$$

the above form turns into Post's and Capel's normalized submap

$$u_{i+1} = |u_i|^z - r \tag{11}$$

First consider the ratios of window widths. If ρ varies by $\Delta\rho$ when μ varies across the period-l window, the window width expressed in μ is obviously

$$\Delta\mu \simeq \Delta\rho \left(\frac{d\rho}{d\mu} \bigg|_{\mu=\tilde{\mu}_l} \right)^{-1}$$

where a tilde on the symbol μ indicates the superstable parameter value. By the scaling, Eq. (10), between ρ and r we can express $\Delta\rho$ in terms of Δr, the invariable normalized window width of Eq. (11). The "physical" window width w_l is then

$$w_l = \Delta\mu \simeq \frac{\Delta r}{|\lambda|^{1/(z-1)}\,(d\rho/d\mu\,|_{\mu=\tilde{\mu}})}$$

We now give the index l the values $N+n$ and N, where $N=m+kn$. For ratios of window widths, Δr cancels, and the scaling in the family becomes asymptotically

$$\hat{\sigma}_k = \frac{w_N}{w_{N+n}} = \left(\frac{\lambda_{N+n}}{\lambda_N} \right)^{1/(z-1)} \frac{d\rho/d\mu\,|_{\mu=\tilde{\mu}_{N+n}}}{d\rho/d\mu\,|_{\mu=\tilde{\mu}_N}} \tag{12}$$

when $N=m+nk$ becomes large, i.e., k becomes large ($\hat{\sigma}_k$ is essentially the product of v subsequent σ_k's defined in Section 5).

For the scaling of the positions of the windows we turn to the function ρ_{m+kn}. It determines the height of the maximum or minimum of the local mapping $f_\mu^{m+kn}(x)$ and it gives the value of x to which the center point $x=0$ is sent in $f_\mu^{m+kn}(x)$. At $\mu=\tilde{\mu}_{m+nk}$, $\rho_{m+nk}=\rho(\mu)$ takes the value zero. Let us look at the functions ρ_{m+nk} for different k at the accumulation point μ_∞ of the family. Denote again $N=m+nk$. From the definition of ρ above, Eq. (9),

$$\rho_N(\mu) = f_\mu^N(0)$$

and

$$\rho_{N+n}(\mu) = f_\mu^n(\rho_N)$$

But the orbit of the center $x = 0$ repeats itself with the period n. It follows that

$$\rho_{N+n}(\mu_\infty) \simeq \rho_N(\mu_\infty)$$

which means that the ρ_{m+kn} for different k all meet at the same point at μ_∞. Assuming that the $\rho(\mu)$ are locally straight lines, we get the asymptotic μ scaling from the derivatives of the ρ for different k. Estimate ρ_N and ρ_{N+n} as follows:

$$\rho_N(\mu_\infty) \simeq -\frac{d\rho_N}{d\mu}\bigg|_{\mu=\tilde{\mu}_N} (\tilde{\mu}_N - \mu_\infty)$$

and

$$\rho_{N+n}(\mu_\infty) \simeq -\frac{d\rho_{N+n}}{d\mu}\bigg|_{\mu=\tilde{\mu}_{N+n}} (\tilde{\mu}_{N+n} - \mu_\infty)$$

These two quantities being equal,

$$\frac{\tilde{\mu}_N - \mu_\infty}{\tilde{\mu}_{N+n} - \mu_\infty} \simeq \frac{d\rho_{N+n}/d\mu \mid_{\mu=\tilde{\mu}_{N+n}}}{d\rho_N/d\mu \mid_{\mu=\tilde{\mu}_N}}$$

and

$$\begin{aligned}\hat{\delta}_k &= \frac{\tilde{\mu}_N - \tilde{\mu}_{N+n}}{\tilde{\mu}_{N+n} - \tilde{\mu}_{N+2n}} \simeq \frac{\tilde{\mu}_N - \mu_\infty}{\tilde{\mu}_{N+n} - \mu_\infty} \\ &= \left(\frac{d\rho_{N+n}}{d\mu}\bigg|_{\mu=\tilde{\mu}_{N+n}}\right) \bigg/ \left(\frac{d\rho_N}{d\mu}\bigg|_{\mu=\tilde{\mu}_N}\right) \end{aligned} \tag{13}$$

again for large N. We have not seen this result in the literature.

What remains is calculating the scaling factors λ and the ratio of $(d\rho_{N+n}/d\mu)$ at $\mu = \tilde{\mu}_{N+n}$ and $d\rho_N/d\mu$ at $\mu = \tilde{\mu}_N$. We follow Post and Capel.[12]

Take λ first. In the following, a tilde on the variable x refers to an iterate of the central value $x = 0$. At superstability, since ρ vanishes there, we have from the local description of Eq. (8)

$$x_{l(i+1)} = \lambda\, |x_{li}|^z \tag{14}$$

where some starting point x_{li} close to the central value $x=0$ has been picked. We will express the left-hand side of this equation as an expansion in terms of the following quantity:

$$v = f_{\bar{\mu}_l}(x_{li}) - f_{\bar{\mu}_l}(\tilde{x}_0)$$

with the same point x_{li}. In a single shot of the mapping $f_{\bar{\mu}_l}$, the central point and a point in its vicinity will be sent far away from the center but roughly to the same location. Therefore the quantity v is small. To linear order in v, then,

$$\begin{aligned} x_{l(i+1)} &= f_{\bar{\mu}_l}^{l-1}(f_{\bar{\mu}_l}(x_{li})) = f_{\bar{\mu}_l}^{l-1}(f_{\bar{\mu}_l}(\tilde{x}_0) + v) \\ &= f_{\bar{\mu}_l}^{l-1}(f_{\bar{\mu}_l}(\tilde{x}_0)) + f_{\bar{\mu}_l}^{(l-1)'}(f_{\bar{\mu}_l}(\tilde{x}_0))v = \prod_{j=1}^{l-1} f'_{\bar{\mu}_l}(\tilde{x}_j)v \end{aligned} \tag{15}$$

where a prime denotes a derivative with respect to the argument of the function at the indicated value of the argument. For the specific form $f(x) = \mu - |x|^z$

$$v = -|x_{li}|^z$$

and Eq. (15) yields

$$x_{l(i+1)} = -\prod_{j=1}^{l-1} f'_{\bar{\mu}_l}(\tilde{x}_j)\,|x_{li}|^z$$

and one reads from Eq. (14)

$$\lambda = -\prod_{j=1}^{l-1} f'_{\bar{\mu}_l}(\tilde{x}_j) \tag{16}$$

With λ_l cleared, one still needs to calculate the derivative of ρ with respect to μ in Eqs. (12) and (13). Look again at the reduced map $f_\mu^l(x)$,

$$x_l = \rho + \lambda\,|x_0|^z$$

and its derivative with respect to μ,

$$\frac{dx_l}{d\mu} = \frac{d\rho}{d\mu} + \frac{d}{d\mu}\lambda\,|x_0|^z$$

Choose x_0 as $\tilde{x}_0 = 0$ and take the expression at the superstable point. The second term on the right vanishes and

$$\left.\frac{d\tilde{x}_l}{d\mu}\right|_{\bar{\mu}_l} = \left.\frac{d\rho}{d\mu}\right|_{\bar{\mu}_l}$$

One thus needs an expression for $(d\tilde{x}_l/d\mu)_{\tilde{\mu}_l}$. Consider the map

$$x_{i+1} = f_\mu(x_i) = \mu - |x_i|^z$$

In general x_{i+1} depends on x_0 in addition to μ, which is indicated by the partial derivatives in the sequel

$$\begin{aligned} \frac{\partial x_{i+1}}{\partial \mu} &= 1 - z\,|x_i|^{z-1}\,\mathrm{sgn}\,x_i\,\frac{\partial x_i}{\partial \mu} = 1 + f'_\mu(x_i)\,\frac{\partial x_i}{d\mu} \\ &= 1 + f'_\mu(x_i)[1 + f'_\mu(x_{i-1})[1 + f'_\mu(x_{i-2})[1 + \cdots]]] \end{aligned} \tag{17}$$

Nothing prevents us from choosing $i+1 = l$ and picking the superstable μ_l along with $x_0 = \tilde{x}_0 = 0$. Then the iterates depend only on μ and

$$\begin{aligned} \left.\frac{d\tilde{x}_l}{d\mu}\right|_{\tilde{\mu}_l} &= 1 + f'_{\tilde{\mu}_l}(\tilde{x}_{l-1})[1 + f'_{\tilde{\mu}_l}(\tilde{x}_{l-2})[1 + \cdots[1 + f'_{\tilde{\mu}_l}(\tilde{x}_1)]\cdots]] \\ &= 1 + \sum_{i=1}^{l-1} \prod_{j=i}^{l-1} f'_{\tilde{\mu}_l}(\tilde{x}_j) \end{aligned}$$

or finally

$$\left.\frac{d\rho}{d\mu}\right|_{\tilde{\mu}_l} = 1 + \sum_{i=1}^{l-1} \prod_{j=i}^{l-1} f'_{\tilde{\mu}_l}(\tilde{x}_j)$$

Now return to Eq. (12). From Eq. (16) we get immediately at the limit when $N \to \infty$

$$\lambda_{N+n}/\lambda_N = \prod_{i=K}^{K+n} f'_{\tilde{\mu}_{N+n}}(\tilde{x}_i)$$

where K is some number larger than M_i, the number of initial iterations before hitting the unstable period, and smaller than $N - M_f$, M_f being the number of the last steps drifting away from the unstable period to end at the center. It is important to notice for what follows that K may vary over the whole middle range of the $N+n$ cycle. It turns out to be convenient to call the individual derivatives in the product L_i and write L for the product from $i = K$ to $i = K+n$. The contribution to $\hat{\sigma}_k$ of the first factor in Eq. (12) is then

$$\left(\frac{\lambda_{N+n}}{\lambda_N}\right)^{1/(z-1)} = L^{1/(z-1)} \tag{18}$$

The second quantity in Eq. (12),

$$\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_{N+n}}\right)\bigg/\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_N}\right) \tag{19}$$

has a structure which is easiest grasped looking at one derivative at a time. In the following the products are written in the order of increasing length, more or less as in Eq. (17). We need clarifying notation:

$$1+L_{N-1}+L_{N-1}L_{N-2}+L_{N-1}L_{N-2}L_{N-3}+\cdots+e=E$$

where $e=L_{N-1}L_{N-2}L_{N-3}\cdots L_{N-M_f}$. Then

$$\begin{aligned}\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_N} &= 1+L_{N-1}+L_{N-1}L_{N-2}+L_{N-1}L_{N-2}L_{N-3}+\cdots\\ &\quad + L_{N-1}L_{N-2}\cdots L_1\\ &= E+e(L_{N-M_f-1}+L_{N-M_f-1}L_{n-M_f-2}+\cdots\\ &\quad + L_{N-M_f-1}L_{N-M_f-2}\cdots L_{N-M_f-n+1}+L)\\ &\quad + eL(L_{N-M_f-n-1}+L_{N-M_f-n-1}L_{N-M_f-n-2}+\cdots\\ &\quad + L_{N-M_f-n-1}L_{N-M_f-n-2}\cdots L_{N-M_f-2n+1}+L)+\cdots\end{aligned} \tag{20}$$

Now the expressions in the parentheses in Eq. (20) are identical since they run over full unstable periods of length n. We again introduce a new symbol B, this time for the sum in the parentheses multiplied by e. Then

$$\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_N} = E+B+BL+BLL+\cdots+BL^{(N-M_f-M_i)/n}+P \tag{21}$$

where

$$P=eL^{(N-M_f-M_i)/n}(L_{M_i-1}+L_{M_i-1}L_{M_i-2}+\cdots+L_{M_i-1}\cdots L_1)$$

The quantity M_f may be thought of as having been chosen such that $(N-M_f-M_i)/n$ is an integer. P represents the ingoing steps before hitting the unstable period.

Introduce yet another symbol $S=S(N)$ for the right-hand side of Eq. (21) without the term E. Remember that the ratio we are seeking to calculate, Eq. (19), is between two expressions of type (21) with the numerator having $N+n$ in the place of the denominator's N. If the denominator is expressed as in Eq. (21), the numerator cycle is longer by n and the right-hand side of Eq. (21) will have one more term,

$$BL^{(N+n-M_j-M_i)/n}$$

and the factor P becomes multiplied with L. This can be interpreted as multiplying by L the terms which were given the name S above and adding back a B. It follows that

$$\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_{N+n}}\right)\bigg/\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_N}\right)=\frac{E+B+LS}{E+S} \tag{22}$$

S is obviously a rapidly growing function of N provided that L is larger than unity, i.e., that the periodic orbit at the accumulation point of the family is unstable. All other quantities in Eq. (22) remain constant when N grows. The asymptotic result at the limit $N \to \infty$ is simply

$$\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_{N+n}}\right)\bigg/\left(\frac{d\rho}{d\mu}\bigg|_{\mu=\bar{\mu}_N}\right)=L$$

By Eq. (13) this result is directly the parameter scaling factor δ,

$$\delta = L$$

Together with Eq. (18) we get for the scaling of the window width, Eq. (12),

$$\sigma = L^{1/(z-1)}L = L^{z/(z-1)}$$

7. DISCUSSION

It was shown in Section 6 that typical binary codes with periodic tails lead to geometric scaling whatever the order z of the critical point. The situation is very different in the case of the exceptional codes. Our numerical results suggest (see Section 5) that the scaling for an exceptional code with a periodic tail is always superexponential and probably universal with a quadratic maximum. If we let the order of the maximum increase, however, the scaling becomes geometric. For the Fibonacci sequences this has been pointed out by Lyubich.[20] The critical value of the exponent z for the Fibonacci sequences is 2. In other cases it is numerically much harder to pinpoint the exact critical value of the universality class at which the transition from superexponential to geometric scaling takes place. This would be an interesting problem for future studies.

A completely different binary ordering of periodic windows has recently been considered by Zaks.[21] Each infinite sequence of periodic windows is described by a "winding number." He observes a double-exponential scaling for the sequence with inverse golden mean winding number. Furthermore, qualitative properties of the scaling are not affected by changing the order of the critical point. In our binary tree, these windows correspond to the binary codes 0, 01^2, 01^201^3, $01^201^31^5$, $01^201^31^501^8$, $01^201^31^501^81^{13}$,..., which are exponentially growing truncations of the infinite aperiodic code $01^201^301^801^{21}$.... Along this binary path the MSS length grows arithmetically resulting in an average geometric scaling of the periodic windows. Therefore, it is not surprising that the scaling is superexponential for the above subsequence.

A binary tree of stable periodic attractors appears also in invertible circle maps.[22] It is therefore interesting to compare the scaling behaviors in these two cases. In a circle map the period of the attractor increases geometrically for almost all routes in the binary Farey tree. In the standard case this leads to universal geometric scaling. In the unimodal map a geometric increase of the period and the resulting universal scaling is observed only in some exceptional cases. It is an open question whether one could construct a "nonrenormalizable" circle map with superexponential scaling. It is not clear how the various scaling properties relate to the Lebesgue measure of the parameter values for aperiodic attractors. For a standard critical cubic circle map this measure is zero, whereas for the quadratic unimodal map the measure of aperiodic attrators is positive.

The typical binary codes with periodic tails give a huge number of new Misiurewicz points in a unimodal map (recall that the same structure is repeated within each window). These points form a subset of the set of all Misiurewicz points of the Mandelbrot set.[23] It would be an interesting problem in the theory of numbers to work out whether the typical binary codes with periodic tails (taking into account the self-similarity) give all the Misiurewicz points of a unimodal map. Any additional ones would have to correspond to asymptotically periodic MSS sequences generated by either exceptional binary codes or typical codes with aperiodic tails.

APPENDIX. A TYPICAL BINARY CODE WITH A PERIODIC TAIL LEADS TO A MISIUREWICZ POINT

Let us assume that the binary code has a periodic tail which is neither 0^∞ nor 1^∞. The code can always be written in the form $IJJJ...$, where I and $J = j_1 ... j_K$ are finite binary codes with $j_2 \neq j_1$. Let A be the corresponding infinite MSS sequence.

Lemma. Consider a binary code of the above form with $h_k < H < \infty$ for every k. Then the accumulation point of the corresponding sequence of periodic windows is a Misiurewicz point.

Proof. Because $j_2 = j_1$, Rule 2(b) implies that $m_{N+2+qK} < 2H$ for $q = 0, 1, 2,...,$ where N is the length of I. Furthermore, by Rules 1(b) and 2(b), $m_{k+1} \leqslant m_k + H$. Combining these two results gives an upper bound for m_k: $m_k < (K+1)H$ for $k = N+2$, $N+3,...$. There exists a $p \geqslant N+2$ such that $l(\hat{A}_p) \geqslant (K+1)H$. Because $m_{k+1} < l(\hat{A}_{k+1})$ for $k = p-1$, $p,...,$ it is not necessary to carry out the antiharmonic extensions when determining h_k ($k \geqslant p-1$) in Rule 1(a) or 2(a). In particular, A can be written as the composition $A_{p-1}\beta_{p-1}\{A\}_1^{h_{p-1}}\beta_p\{A\}_1^{h_p}\cdots$. The set of possible values for m_k and h_k is finite when $k > N+1$. Therefore, there exist finite $P > p$ and Q so that $(m_P, h_P, h_{P-1}) = (m_{P+QK}, h_{P+QK}, h_{P+QK-1})$. Because $i_{P+k} = i_{P+QK+k}$ for $k = 0, 1, 2,...,$ Rules 1 and 2 imply that

$$(m_{P+k}, h_{P+k}, h_{P+k-1}) = (m_{P+QK+k}, h_{P+QK+k}, h_{P+QK+k-1})$$

which is possible only if A has a periodic tail. By Corollary II.8.4 of ref. 1, A is an MSS sequence for a Misiurewicz point. ▮

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